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AIR POLLUTION AND URBAN POPULATION DISTRIBUTION

C. Peter Rydell, et al

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AIR POLLUTION AND URBAN POPULATION DISTRIBUTION

C. Peter Rydell

with the collaboration of

Douglas Collins
Andrew Dedrick
Helen Kaminski
Gretchen Schwarz

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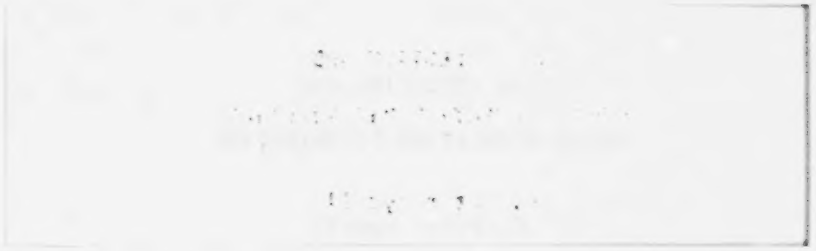
July, 1968



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PREFACE

Inspiration for the following investigation came from Professor Benjamin H. Stevens at the University of Pennsylvania. His interest in the effects of macro urban form on air pollution started us first jointly and later separately on studies of the question.

Douglas Collins helped formulate the matrix approach to population distribution analysis and the pollution diffusion model. Andrew Dedrick wrote and executed the FORTRAN computer programs that performed the many required calculations. Helen Kaminski wrote Appendix A evaluating the assumptions about pollution generation. Gretchen Schwarz wrote Appendix C evaluating the assumptions about pollution damage.

Twice is not too many times to thank the Research Grants Branch, National Center for Air Pollution Control, Bureau of Disease Prevention and Environmental Control, U. S. Public Health Service.

Finally, the constant secretarial support of Mrs. Anne Jones was an invaluable contribution to the completion of this research.

TABLE OF CONTENTS

Preface..... ii

List of Figures..... iv

1. Introduction: Air Pollution and Optimal Urban Form..... 1

2. Population Distribution that Minimizes Damage From Area Sources with Emissions Proportional to Population Density..... 4

3. Population Distribution that Minimizes Damage From Area Sources with Emissions a Linear Transformation of Population Density..... 8

4. Population Distribution that Equalizes Per Capita Air Pollution Damages..... 18

5. Effect of Air Pollution Costs on Optimal Urban Form..... 24

6. Conclusions..... 30

Appendix A:
Air Pollution Generation from Area Sources..... 32

Appendix B:
Air Pollution Diffusion Model..... 42

Appendix C:
Evaluating Air Pollution Damage..... 50

Appendix D:
Explaining Urban Population Distribution by Density Cost Functions..... 60

Appendix E:
Calculating Population Distributions that Minimize a Quadratic Cost Function..... 65

LIST OF FIGURES

1. Urban Area Divided into Numbered Zones..... 4

2. Population Distribution that Minimizes Damage from Air Pollution Generated Proportional to Population Density..... 7

3. Pollution Generation Matrix, G, for Traffic Pattern 1..... 10

4. Pollution Generation Matrix, G, for Traffic Pattern 2..... 11

5. Pollution Generation Matrix, G, for Traffic Pattern 3..... 12

6. Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 1..... 15

7. Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 2..... 16

8. Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 3..... 17

9. Population Distribution that Equalizes the Per Capita Costs from Locally Generated Air Pollution..... 20

10. Population Distribution that Equalizes the Per Capita Costs from Air Pollution under Traffic Pattern 1..... 21

11. Population Distribution that Equalizes the Per Capita Costs from Air Pollution under Traffic Pattern 2..... 22

12. Population Distribution that Equalizes the Per Capita Costs from Air Pollution under Traffic Pattern 3..... 23

13. Per Capita Access Costs by Zone..... 24

14. Population Distribution that Minimizes Access Plus Density Costs.... 27

15. Population Distribution that Minimizes Air Pollution Costs..... 28

16. Population Distribution that Minimizes Total Costs..... 29

Appendix A:

1. Sulfur Oxide Emissions from Burning of Coal, Fuel Oil, and Gas in Residential Heating Units..... 33

Appendix B:

1. Urban Area Divided into Grid Cells..... 42

LIST OF FIGURES

(Cont'd.)

Appendix B: (Cont'd.)

2. Air Pollution Diffusion Matrix, D..... 44

3. Diagram for Equation (6)..... 46

4. Diagram for Equation (7)..... 47

5. Average Concentrations Caused in Downwind Cells by Source Cell, Stated as Fractions of the Average Concentration in the Source Cell..... 48

1. INTRODUCTION: AIR POLLUTION AND OPTIMAL URBAN FORM

Air pollution damage is caused by three variables: the intensity of emission; the transport, diffusion, and sometimes chemical alteration of pollutants by the atmosphere; and the impact of pollutants on people and property. While this trilogy of emission, transmission, and reception gives a correct description of the causes of air pollution, it is an incomplete guide in the search for a cure. Since atmospheric transmission mechanisms and receptor sensitivity are largely uncontrollable, attempts to control air pollution tend to focus on reducing emissions. There is, however, another controllable variable to be considered: urban form. The relationship between emittors and receptors can sometimes be manipulated to reduce air pollution damage without any change in amount of pollutant emitted, meteorological conditions, or receptor sensitivity. A familiar example is the reduction in air pollution damage achieved by locating a smoky factory on the leeward edge of a city instead of on the windward edge. Another example of modifying urban form to reduce air pollution damage is using parks to buffer residences from industrial pollution. (See Rydell and Schwarz [3] for a review of the literature on the relation between urban form and air pollution.)

Point and line sources of air pollution are relatively easy to analyze. The proper spatial relationship between receptors and emittors becomes more complicated when area sources such as residential space heating and automobile traffic are considered. Further complicating the analysis is the fact that space heating and automobile pollution emissions are closely related to population location. So it is not feasible to separate emission and reception by buffer zones.

Thus a current challenge to urban location theory is to design urban forms that minimize the impact of pollutants whose generation cannot be avoided. One can trace this challenge back to 1884 when it was argued by Blasius [1] that if cities in the northern temperate zone are built in a westerly direction "the prevailing air current (south, south-west in summer) will pass over such a built up city from its broad side, and furnish the whole population with pure air from the country". On the other hand "if the city is built up to the north, the prevailing current passes over its entire length, and will furnish the whole population with the impurities generated in the lower parts of the city, accumulating poisons as it advances".

The present paper is the third in a series of attacks on the problem of designing macro urban form to reduce air pollution damage. The two previous approaches to this problem considered the effects of city shape and population density on urban air pollution damage. In Rydell and Stevens [4] it was found that under certain conditions air pollution damage is minimized if urban areas have parabolic boundaries and are elongated in a direction perpendicular to the prevailing wind direction. In Rydell and Collins [2] it was found that lowering the average population density reduces air pollution damage, even though a larger urban areas means increased automobile travel and therefore increased pollutant generation.

Both these previous studies assumed that population density was constant over the urban area. The present analysis explores the previously neglected effect of pollution distribution on air pollution damage. The major conclusion is that redistributing population from high density central zones to low density fringe zones would reduce total air pollution damage

in urban areas. Such a redistribution can accomplish about a 30% reduction in air pollution damage from area sources. However, when non air pollution costs are considered, it turns out that redistribution can only reduce total urban area costs by 1%. (See Section 5) Surprisingly, total urban area costs turn out to be relatively insensitive to population distribution.

REFERENCES

[1] Blasius, William. "In Which Direction Should Cities In Our Latitude Extend In Order to Secure to Their Population Pure Air and Thereby Health?" American Philosophical Society of Philadelphia, 1884.

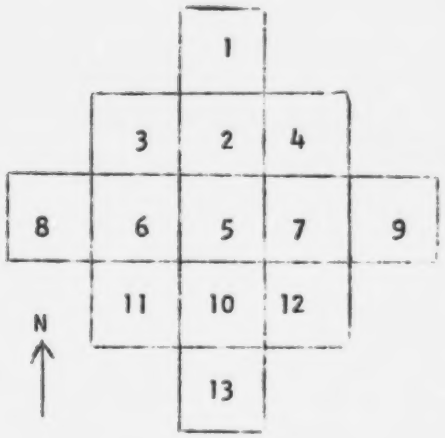
[2] Rydell, C. P. and Collins, D. "Air Pollution and Optimal Urban Form", mimeo, Hunter College Urban Research Center, City University of New York, (June 1967).

[3] Rydell, C. P. and Schwarz, G. "Air Pollution and Urban Form: Review of Current Literature", Journal of the American Institute of Planners, Vol. 34, No. 2, (March 1968) pp. 115-120.

[4] Rydell, C. P. and Stevens, B. H. "Air Pollution and the Shape of Urban Areas", Journal of the American Institute of Planners, Vol. 34, No. 1, (January 1968) pp. 50-51.

2. POPULATION DISTRIBUTION THAT MINIMIZES DAMAGE FROM AREA SOURCES WITH EMISSIONS PROPORTIONAL TO POPULATION DENSITY

Start with an urban area divided into equal sized zones (see Figure 1), number the zones, and then record the counts of zone populations as a column vector, $\underline{p}$. A simple model of air pollution costs caused by households can now be constructed using a matrix transformation of this population



$$\underline{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \\ p_{13} \end{bmatrix}$$

Figure 1: Urban Area Divided into Numbered Zones

vector. Assume that pollution generation due to household heating and incineration and local automobile trips is directly proportional to population density. Let the matrix D be the transformation that represents pollutant diffusion. Then $D\underline{p}$ is proportional to the vector of pollutant concentrations. If we now make the assumption that the cost of pollution damage to an individual household is directly proportional to pollutant concentration, then the total cost of air pollution damage is proportional to the quadratic form: $\underline{p}'\, D\, \underline{p}$. These assumptions about air pollution generation, diffusion, and impact are discussed in Appendices A, B, and C.

The population distribution that minimizes air pollution damage is the vector $\underline{p}$ that minimizes the quadratic form $\underline{p}'\ D\ \underline{p}$ subject to the nonnegativity constraint $\underline{p} \geq \underline{0}$. This minimization is also subject to the condition that all population be distributed, $\underline{j}'\ \underline{p} = t$; where $\underline{j}$ is a column vector of ones and t is the total population.

This minimization problem is a quadratic programming problem:

Choose $\underline{p}$

to minimize $\underline{p}'\ D\ \underline{p}'$

Subject to $\underline{p} \geq \underline{0}$ and $\underline{j}'p = t$.

It can be solved by the classical Lagrange multiplier method as long as the nonnegativity constraints do not bind:

Choose $\underline{p}$ and λ

to minimize $\underline{p}'\ D\ \underline{p} + \lambda[t - \underline{j}'\ p]$

To be able to use the theorem: where A is a symmetric matrix the derivative of $\underline{p}'\ A\ \underline{p}$ with respect to the vector $\underline{p}$ is $2\ A\ \underline{p}$, (Morrison [1] p. 68), the above quadratic form must be rewritten with a symmetric matrix:

Choose $\underline{p}$ and λ

to minimize $\underline{p}' \left[\frac{D' + D}{2}\right] \underline{p} + \lambda\ [t - \underline{j}'\underline{p}]$

The first order conditions for a minimum are found by differentiating with respect to the vector $\underline{p}$:

$$2\left[\frac{D' + D}{2}\right]\ \underline{p} = \lambda\ \underline{j}$$

Solving for $\underline{p}$ and using $\underline{j}'\underline{p} = t$ to evaluate λ the population distribution that minimizes air pollution damage is

$$(1) \quad \underline{p} = \frac{t\,[D' + D]^{-1}\,\underline{j}}{\underline{j}'\,[D' + D]^{-1}\,\underline{j}}$$

EXPERIMENTAL RESULTS

Appendix B presents the specific diffusion matrix, D, used in this research's experiments. (See Figure 2, Appendix B.) It happens that the matrix $[D' + D]/2$, in the quadratic form being minimized, has a dominant diagonal. Appendix E shows that this dominant diagonal not only guarantees the second order conditions for a minimum, but also guarantees that the non-negativity constraints will not bind.

Using the diffusion matrix in Appendix B and setting the population total at one million, $t = 1{,}000{,}000$, equation (1) gives an almost uniform population distribution. (See Figure 2.) This result could have been anticipated from the dominant diagonal in the diffusion matrix which makes most of the pollution damage occur in the originating grid cell, thus leading to a uniform solution for all grid cells. The conclusion suggested by this initial experiment is that air pollution damage can be reduced by redistributing population from high density central districts to the low density fringes of cities.

The damage cost of 83.9, given in Figure 2, is the value of $\underline{p}'\,D\,\underline{p}$ divided by 10^9. This cost number can be compared with the cost numbers in subsequent experiments, but no attempt has been made in this research to evaluate air pollution damage in absolute terms.

REFERENCE

[1] Morrison, Donald F. MULTIVARIATE STATISTICAL METHODS, McGraw-Hill, 1967.

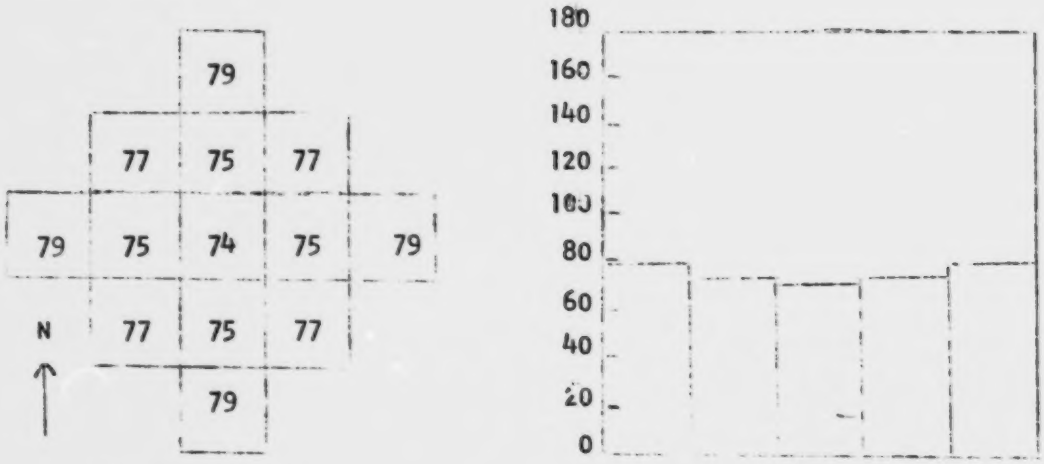


Fig. 2: Population Distribution that Minimizes Damage from Air Pollution Generated Proportional to Population Density. Total Damage = 83.9

3. POPULATION DISTRIBUTION THAT MINIMIZES DAMAGES FROM AREA SOURCES WITH EMISSIONS A LINEAR TRANSFORMATION OF POPULATION DENSITY

The previous section only considered air pollution from the area sources of household heating and incineration and local automobile traffic. Long distance automobile trips, however, generate pollutants in other grid cells besides the origin grid cell. This section generalizes the matrix model to make pollution generation a non-diagonal matrix transformation of population density. The objective function of minimizing air pollution damage remains a quadratic form. The Lagrange multiplier technique also remains applicable, for although there are no longer theoretical guarantees of positive definite matrices and non-negative solutions to the Lagrange problem, these requirements are met in all experiments.

NOTATION

$\underline{p}$ = vector of population counts in each grid cell of an urban area.

t = total population of urban areas, $\underline{j}' \underline{p} = t$ where $\underline{j}$ is a column vector of ones.

$\underline{h}$ = vector of pollutant emissions from households and local traffic.

$\underline{a}$ = vector of pollutant emissions from automobile trips to city center.

$\underline{c}$ = vector of air pollution concentrations.

ASSUMPTIONS

1) $\underline{c} = D[\underline{h} + \underline{a}]$, the vector of air pollution concentrations is a linear transformation of pollutant emissions. The matrix, D, describes pollutant distribution by wind and diffusion. (See Appendix B.)

2) $\underline{h} = m\,\underline{p}$, the vector of pollutant emissions from households and local automobile traffic are proportioned to population density.

3) $\underline{a} = n\,G\,\underline{p}$, the vector of pollutant emissions from automobiles traveling to central district are proportional to a linear transformation of population density. This matrix, G, describes the pattern of travel to center generated by households located in the different grid cells.

DAMAGE MINIMIZING POPULATION DISTRIBUTION

Putting the three assumptions together, the total cost of air pollution, $\underline{p}'\underline{c}$, is seen to equal $\frac{1}{2}\,\underline{p}'\,A\,\underline{p}$ where $A = D[mI + nG] + [mI + nG]'\,D'$. Appendix E shows that this quadratic form has a minimum, subject to $\underline{j}'\underline{p} = t$, if either A or $[a_{11}\,\underline{j}\,\underline{j}' - 2\,\underline{a}\,\underline{j}' + A_{22}]$ is a positive matrix; and that the minimizing population distribution is given by:

$$p = \frac{t\,A^{-1}\,\underline{j}}{\underline{j}'\,A^{-1}\,\underline{j}} \tag{2}$$

GENERATION MATRICES FOR AUTOMOBILE POLLUTION

In this section's experiments three alternative traffic patterns for non-local automobile trips are explored. In all patterns, trips that leave the origin grid cell are assumed to travel over a grid system of arterials and to terminate in the central grid cell of the urban area (cell 5 in Figure 1). Pattern I assumes that the average trip to the city center travels one-half the width of the origin and destination cells and the entire width of the intermediate cells. Fifty percent of the traffic from cell 3 to cell 5 travels through cell 2 and fifty percent travels through cell 6. The same 50-50 split of traffic over a grid street system is assumed for traffic from cells 4, 11, and 12. Figure 3 gives the matrix, G, that relates automobile pollution generation to population density under this first traffic pattern.

From Trip - Origin Zone

To Traffic Zone	1	2	3	4	5	6	7	8	9	10	11	12	13
1	.5												
2	1.0	.5	.5	.5									
3			.5										
4				.5									
5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
6			.5			.5		1.0			.5		
7				.5			.5		1.0			.5	
8								.5					
9									.5				
10										.5	.5	.5	1.0
11											.5		
12												.5	
13													.5

Figure 3: Pollution Generation Matrix, G, for Traffic Pattern 1.

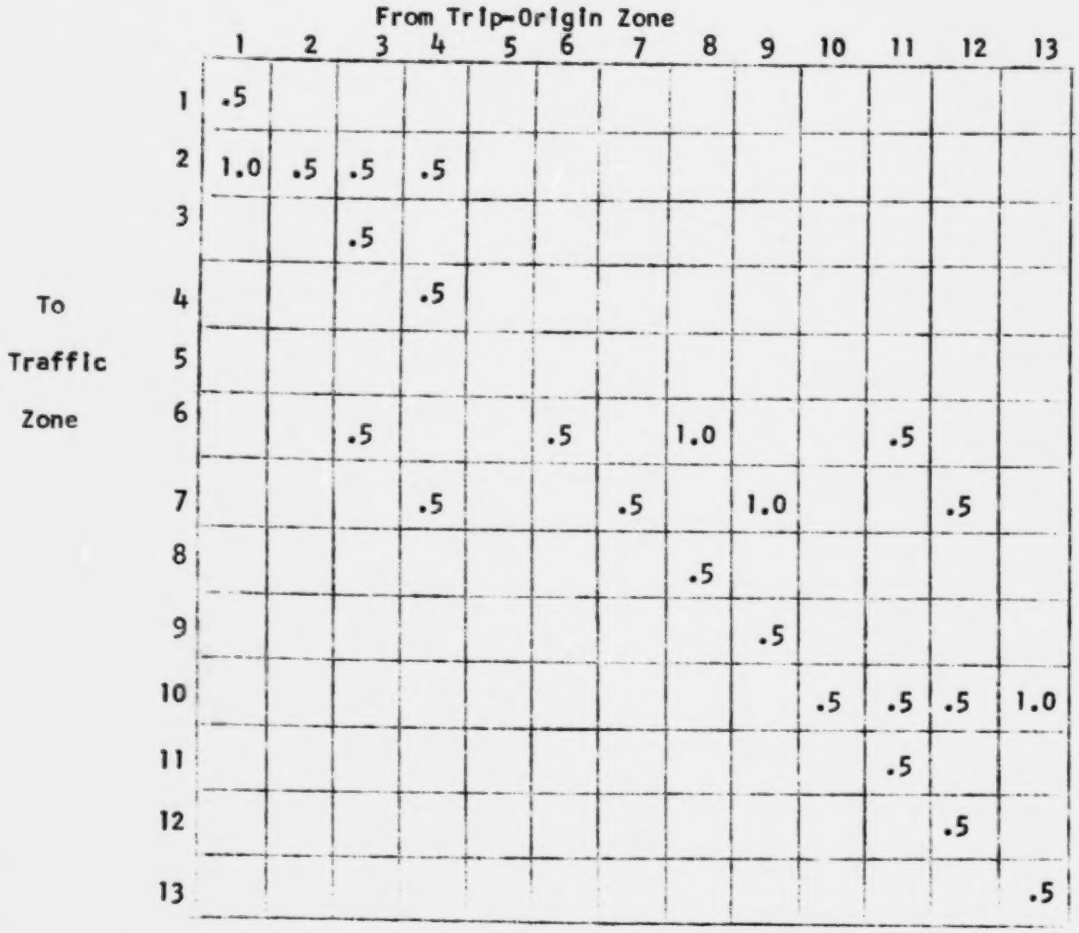


Figure 4: Pollution Generation Matrix, G, for Traffic Pattern 2 (No Cars in Central Zone.)

From Trip-Origin Zone

To Traffic Zone	1	2	3	4	5	6	7	8	9	10	11	12	13
1	.5												
2	1.0	.5	1.0	1.0									
3			.5										
4				.5									
5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
6						.5	.	1.0					
7							.5		1.0	1.0			
8								.5					
9									.5	.			
10										.5	1.0	1.0	1.0
11											.5	.	
12												.5	
13													.5

Figure 5: Pollution Generation Matrix, G, for Traffic Pattern 3 (North-South Expressway)

The second and third traffic patterns are variations on the first. Traffic Pattern Number 2 assumes that cars have been banned from the central district of the urban area. Commuters are assumed to park their cars at the edge of the central district and complete their trip by electric cars, pollution free busses, walking, etc. This assumption of no pollution emission in the central cell is embodied in the G matrix in Figure 4.

The third traffic pattern assumes that a north-south expressway has been constructed. This expressway alters traffic flows so that all traffic from cells 3 and 4 to cell 5 now goes through cell 2. Similarly, all traffic from cells 11 and 12 to cell 5 is now assumed to go through cell 10. Figure 5 presents the generation matrix, G, that contains the expressway assumption.

EXPERIMENTAL RESULTS

In the following experiments m = 1.00 and n = 0.18. These constants were arbitrarily chosen to reflect a reasonable balance between "household" and "work trip" pollutant sources. If n is chosen much larger than 0.18, population in the central square is forced to zero in the damage minimizing population distribution.

The population distribution that minimizes air pollution damage, $\frac{1}{2}\,\underline{p}\,A\,\underline{p}$, under traffic pattern 1 turns out to be a cup shaped distribution. (See Figure 6.) This is not surprising, since the household pollution sources led to a near uniform cost minimizing form and the addition of central traffic pollution should force more central population to be relocated.

With a ban on cars in the central zone, the center becomes a better place to locate population, and the pollution cost minimizing form is more like a cup with a raised center. (See Figure 7.) The expressway traffic pattern causes a slight population shift away from the expressway route. (See Figure 8.)

The air pollution damage numbers in Figures 6, 7, and 8 are $\frac{1}{2}\,\underline{p}'\,A\,\underline{p}$ divided by 10^9. As mentioned earlier these numbers have not been given an absolute interpretation, but only a relative one. The damage cost of 107 for traffic pattern 1 shows the effect of adding work trip pollution. In Figure 2, with no work trips, air pollution damage was only 83.9.

When cars are prohibited in the central zone, total damage falls to 100. The installation of the expressway has no aggregate effect on air pollution costs.

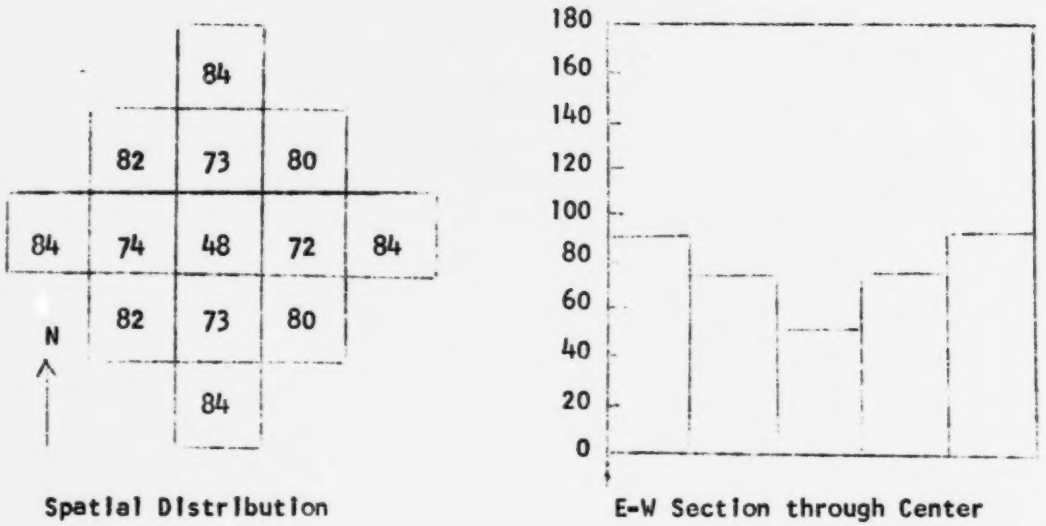
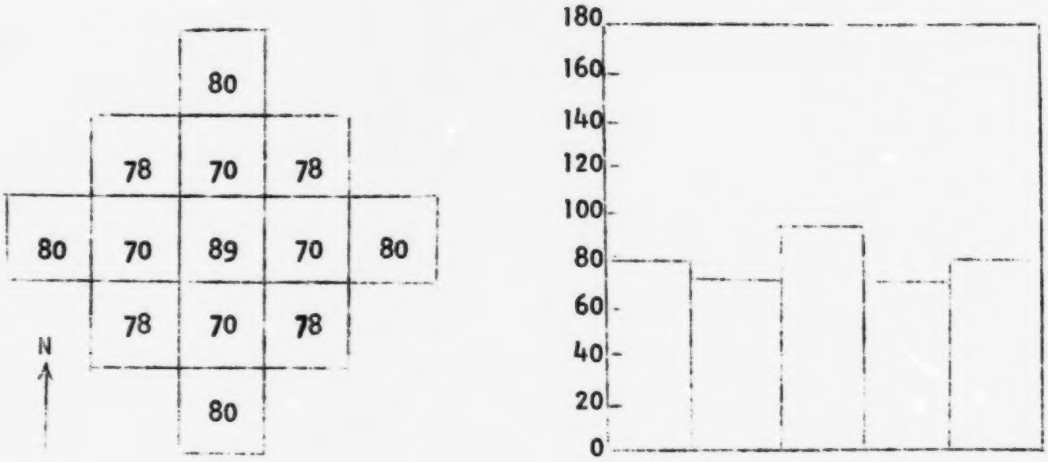


Figure 6: Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 1. Total Damage = 107.

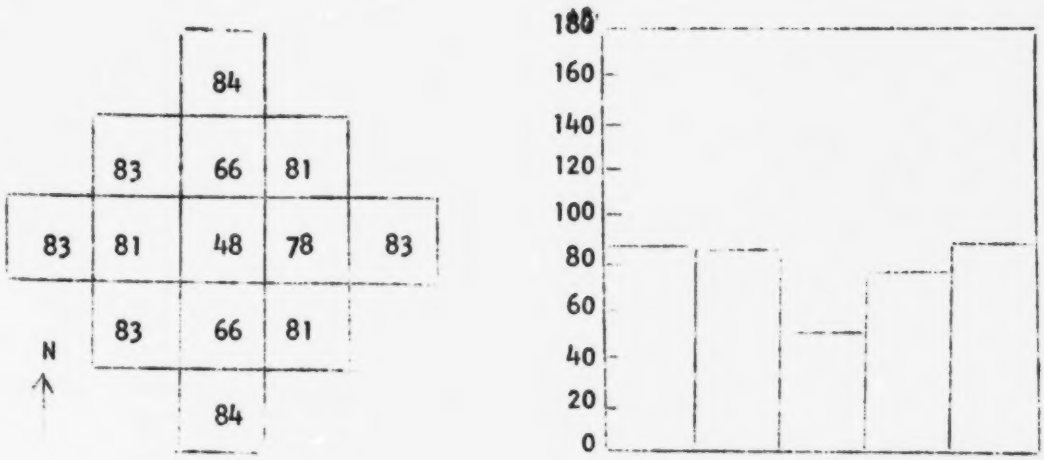


Spatial Distribution

E-W Section through Center

Population Counts in 1000's

Figure 7: Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 2 (No Cars in Central Zone). Total Damage = 100.



Spatial Distribution

E-W Section through Center

Population Counts in 1000's

Figure 8: Population Distribution that Minimizes Air Pollution Damage from Area Sources, with Traffic Pattern 3 (North-South Expressway). Total Damage = 107.

4. POPULATION DISTRIBUTION THAT EQUALIZES PER CAPITA AIR POLLUTION DAMAGES

In contrast to the population distribution that minimizes total air pollution cost, is the distribution that results if each person individually seeks to minimize air pollution concentration in the air around his home. This equilibrium of private action will be reached when all zones of the urban area have equal pollutant concentration. The equilibrium will be stable because an increase in a zone's population would result in an increase in pollution concentration, consequently any one who is moved from his equilibrium position will encounter increased pollution and want to restore the equilibrium pattern by moving back.

The vector of pollution concentrations is given by $\underline{c} = E\,\underline{p}$ where $E = D[mI + nG]$. Concentrations will be the same in all zones if $E\,\underline{p} = \lambda \underline{j}$ where $\underline{j}$ is a column vector of ones, and λ is the multiplier that makes $\underline{j}'\,\underline{p} = t$. Solving these equations yields $\underline{p} = E^{-1}[\lambda \underline{j}]$ and $\lambda = t/\underline{j}'\,E^{-1}\,\underline{j}$, so the equilibrium distribution of population is:

$$p = \left[\frac{t}{\underline{j}'\,E^{-1}\,\underline{j}}\right] E^{-1}\underline{j} \tag{3}$$

This equation has an identical form with the equation (2) defining optimal population distribution, which simplifies numerical calculations. The two equations can be compared by recalling that $A = E' + E$.

MAKING EQUILIBRIUM OPTIMAL BY TAXATION

Add a tax vector, $\underline{x}$, where x_i is the tax paid by each inhabitant of zone i, to the air pollution cost vector, $\underline{c}$; $\underline{c} + \underline{x} = E\,\underline{p} + \underline{x}$. The problem is to find the tax plan that will make the equilibrium population distribution, arrived at by people seeking the lowest personal cost location, the same as

the optimal population distribution, so that total air pollution cost is minimized. Subject to $\underline{j}' \underline{p} = t$ the optimal distribution must maximize $\frac{1}{2} \underline{p}' [E' + E] \underline{p} + \underline{p}'\underline{b}$, which means that $[E' + E] \underline{p} = \lambda \underline{j}$ must be satisfied. The equilibrium distribution must satisfy $E \underline{p} + \underline{x} = \lambda \underline{j}$. For these two equations to have the same solution it is necessary that $\underline{x} = E' \underline{p}$.

This tax plan, $\underline{x} = E' \underline{p}$, can be interpreted as charging each person with the air pollution damage he causes others, thus internalizing the formerly external costs of his location decision.

EXPERIMENTAL RESULTS

The major conclusion that can be drawn from the following equal-per-capita-cost urban forms, Figures 9 - 12, is that they are not very different from the minimum total-air-pollution-cost urban forms discussed in the previous sections. That the wind in this analysis is assumed to come from the northwest and southwest explains the tendency for the equilibrium distribution to concentrate population in the west. The total costs are all larger than the corresponding minimum costs, as they have to be, but not as much larger as was expected.

The fact that the equilibrium distributions and total costs are not very different from the corresponding minimum cost solutions indicates that the externalities associated with area sources are not as significant as one would have thought. This conclusion implies that informing individual locators of the air pollution costs associated with different locations might be all that is necessary to achieve a nearly optimal population distribution as far as air pollution damage from area sources is concerned.

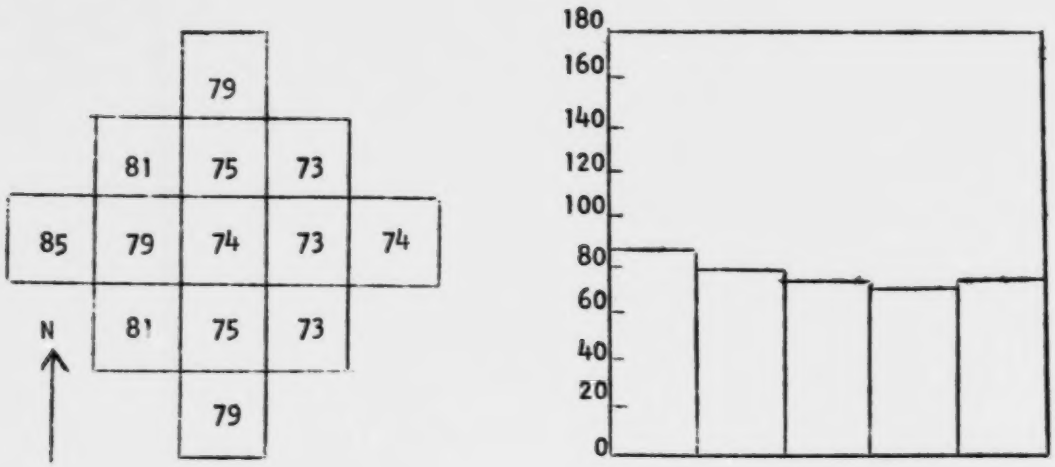
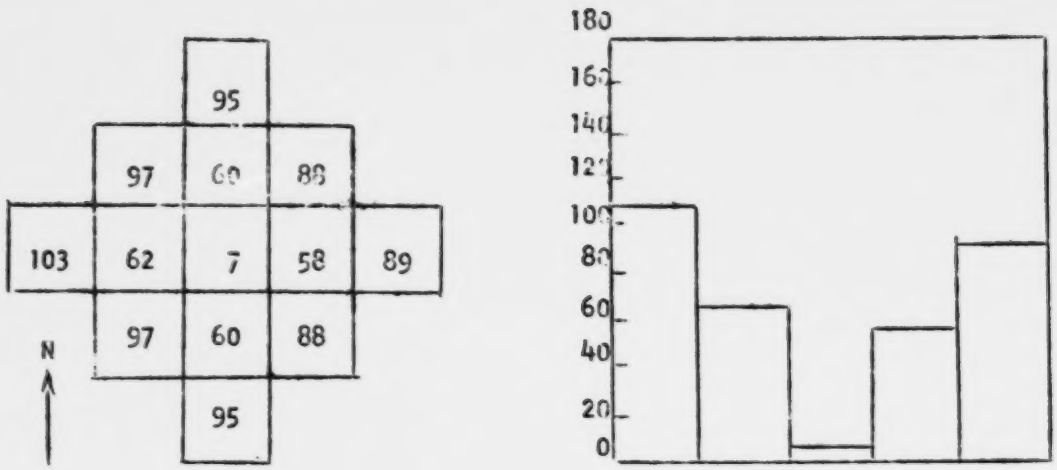


Figure 9: Population Distribution that Equalizes the Per Capita Costs from Locally Generated Air Pollution. Total Damage = 84. Compare with Figure 2.



Spatial Distribution

E-W Section through Center

Population Counts in 1000's

Figure 10: Population Distribution that Equalizes the Per Capita Costs from Air Pollution under Traffic Pattern 1. Total Damage = 111. Compare with Figure 6.

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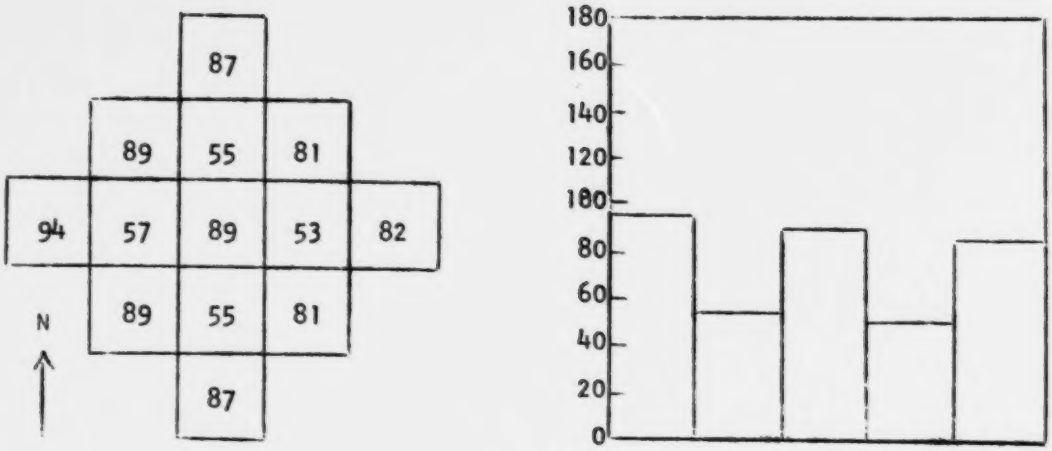


Spatial Distribution

E-W Section through Center

Population Counts in 1000's

Figure 11: Population Distribution that Equalizes the Per Capita Costs from Air Pollution Under Traffic Pattern 2 (No Cars in Central Zone). Total Damage = 101. Compare with Figure 7.

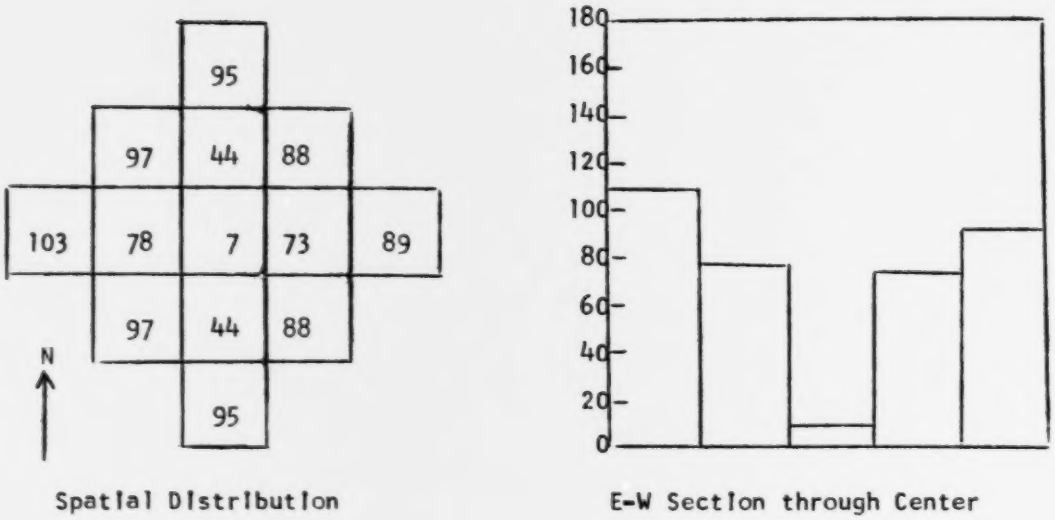


Figure 12: Population Distribution that Equalizes the Per Capita Costs from Air Pollution Under Traffic Pattern 3 (North-South Expressway). Total Damage = 111. Compare with Figure 8.

5. EFFECT OF AIR POLLUTION COSTS ON OPTIMAL POPULATION DISTRIBUTION

In order to assess the importance of population distribution as a method of reducing air pollution damage, the total costs of urban population distributions must be looked at. In the simple model employed in this research these total costs consist of access costs, density or congestion costs, and air pollution costs.

The cost of access to the center of the urban area increases with distance from the center. Figure 13 shows the per capita access cost numbers assumed in the following experiments.

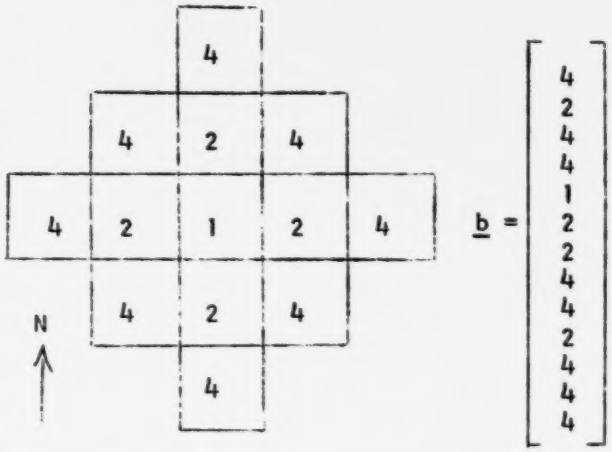


Figure 13: Per Capita Access Costs by Zone

Where $\underline{p}$ is the population distribution vector and where $\underline{b}$ is the vector of per capita access costs in Figure 13, total access cost is proportional to $\underline{b}' \underline{p}$.

Following the analysis in Appendix D, assume that preferences for low density and building construction costs are all reflected in a per capita density cost that is proportional to population density in each zone. The

density cost for the urban area is therefore proportional to $\underline{p}'\underline{p}$.

Air pollution costs, as in section 3, are assumed to consist of local pollution costs, $\underline{p}'\underline{D}\underline{p}$, and central pollution costs, $\underline{p}'\underline{D}\underline{G}\underline{p}$, where D is the diffusion matrix and G is the first traffic pattern pollution generation matrix. "Local pollution" is generated by households and neighborhood automobile trips. "Central pollution" is generated by automobile travel to the central zone.

WEIGHTING THE DIFFERENT COSTS

The two types of air pollution costs are arbitrarily combined as $\underline{p}'\underline{D}\underline{p} + [.18]\ \underline{p}'\underline{D}\underline{G}\underline{p}$, as they were in section 3.

In order to combine access and density costs with the air pollution costs, access costs were weighted by 3.78×10^5, and density costs were weighted by 3.94. Thus the total cost of a population distribution is assumed proportional to

$$[3.78 \times 10^5]\ \underline{b}'\underline{p} + [3.94]\ \underline{p}'\underline{p} + \underline{p}'\underline{D}\underline{p} + [.18]\ \underline{p}'\underline{D}\underline{G}\underline{p}$$

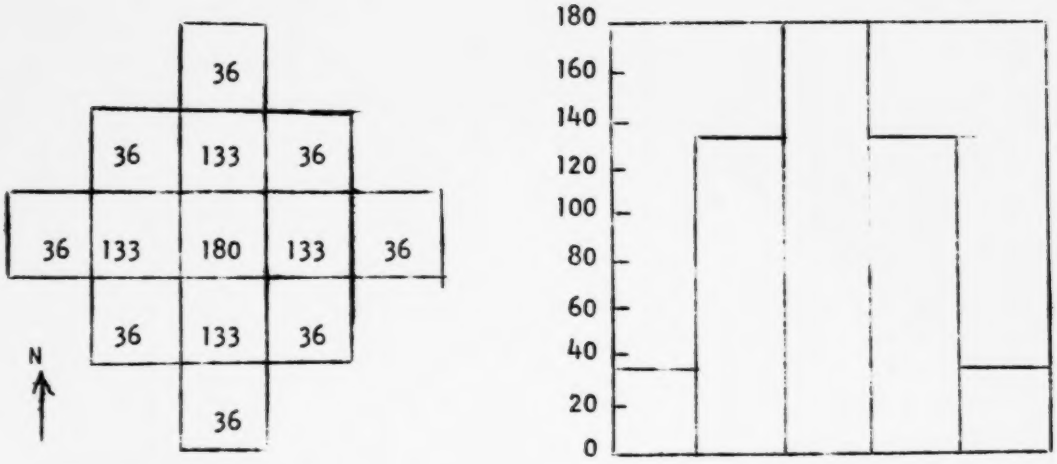
These weights were chosen to make the urban form that minimizes the sum of access plus density costs have a central population density five times the fringe population density. This ratio approximates that currently found in cities of one million population. The weighting numbers were also chosen to make the air pollution cost approximately ten percent of total cost. This seemed as appropriate an assumption as any, given the current difficulties in assessing the costs of air pollution damage. (See Appendix C.)

EXPERIMENTAL RESULTS

Using Appendix E to indicate the required calculations one eventually arrives at Figures 14 - 15 which show the population distribution resulting

from respectively minimizing access plus density costs, pollution damage costs, and total costs. All costs in the Figures have been divided by 10^9. The minimum total cost distribution (Figure 16) is a compromise between the centrally peaked distribution (Figure 14) that minimizes access plus density costs and the shallow cupped distribution (Figure 15) that minimizes air pollution costs. However, the total cost of the optimal compromise is 1502 which is much closer to the total cost, 1511, of the minimum access plus density cost form, than to the total cost, 1669, of the minimum air pollution cost form. This is a reasonable result since air pollution was assumed to be only ten percent of total cost.

The fact that the optimal urban form with air pollution costs considered is only one percent less costly than the optimal urban form when air pollution costs are not considered, means that redistributing population is not as powerful a method of reducing air pollution damage as one could wish. However, air pollution costs do drop from 155 to 132 (about half way to the minimum of 107 achievable by redistributing population) when air pollution costs are considered in planning the population distribution.



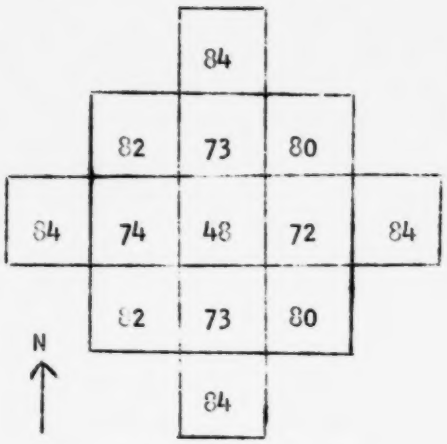
Spatial Distribution

E-W Section through Center

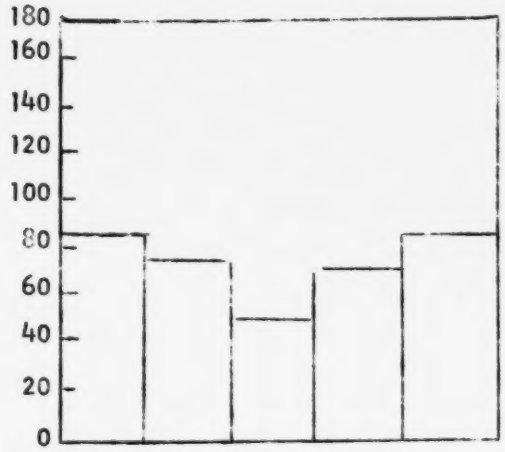
Population Counts in 1000's

Access Cost	Density Cost	Local Air Pollution Cost	Central Air Pollution Cost	Access Plus Density Costs	Total Air Pollution Costs	Total Cost
904	452	122	33	1356	155	1511

Figure 14: Population Distribution that Minimizes Access Plus Density Costs.



Spatial Distribution

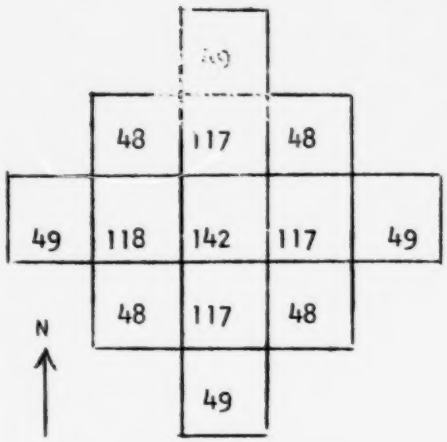


E-W Section through Center

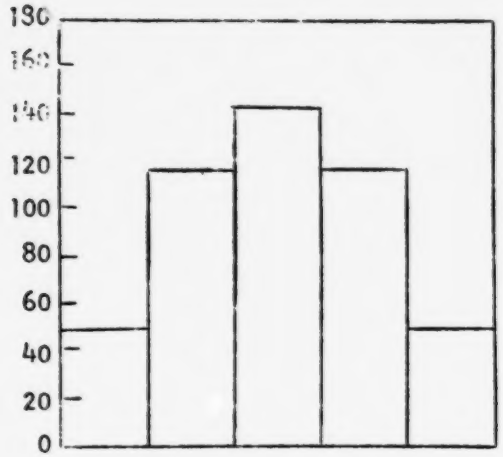
Population Counts in 1000's

Access Cost	Density Cost	Local Air Pollution Cost	Central Air Pollution Cost	Access Plus Density Costs	Total Air Pollution Costs	Total Costs
1244	318	85	22	1562	107	1669

Figure 15: Population Distribution that Minimizes Air Pollution Costs.



Spatial Distribution



E-W Section through Center

Population Counts In 1000's

Access Cost	Density Cost	Local Air Pollution Cost	Central Air Pollution Cost	Access Plus Density Costs	Total Air Pollution Costs	Total Costs
995	375	102	30	1370	132	1502

Figure 16: Population Distribution that Minimizes Total Costs.

6. CONCLUSIONS

When air pollutant generation is assumed to be proportional to population density, a nearly uniform urban population distribution minimizes pollution damage. (See Figure 2.) However, only household heating and incineration and local automobile travel can reasonably be assumed to generate pollutants in proportion to population density. When work and shopping trips to center city are also considered, a cup-shaped population distribution minimizes air pollution damage. (See Figure 6.) If automobiles are banned from the central city area, then the approximately uniform population distribution again minimizes pollution damage. (See Figure 7.) When other costs are considered as well as air pollution costs, the optimal population distribution is a mildly-peaked compromise between the roughly uniform distribution that minimizes air pollution costs and the strongly-peaked distribution that minimizes the other costs. (See Figures 14-16.)

The maximum amount of air pollution damage reduction achievable by population redistribution is about 30%. However, to achieve this reduction would require a cup-shaped distribution for which transportation costs would be excessive. The more appropriate objective of minimizing all costs, not just air pollution costs, achieves less reduction in air pollution damage - only about 15%. In addition, this 15% reduction in air pollution damage must be paid for by a 1.7% increase in the non-pollution costs of access and density. (Percentages calculated from Figures 14 and 16.) Because of the assumption that current air pollution damage is approximately 10% of total urban area costs, the 15% cost decrease and the 1.7% cost increase nearly cancel leaving a net reduction in total urban costs of only 1%. Within the assumptions made in this exploratory

analysis, the conclusion is that urban area costs can not be greatly reduced by population redistribution.

These conclusions are dependent on a host of assumptions which, if not met, necessitate modification of the analysis. For instance, a most blatant polluter of the environment, the industrial sector, has not been considered in this paper in order to focus on area sources of air pollution. The effect of point sources would be to produce pockets of low density around those sources in an optimal population distribution. Secondly, photochemical reactions have not been explored. If air pollution damage increases sharply after prolonged exposure of pollutants to sunlight the diffusion matrix would tend to lose its dominant diagonal feature since damage would occur only after the pollutant had been transported from its origin zone. This, in turn, would make meeting the second order condition of the cost minimization problem less likely and open up the possibility that a discontinuous population distribution would be optimal. Finally, the assumptions that pollution generation is a linear transformation of population densities and that damage is proportional to concentration are only a first approximation to the facts, as Appendices A and C demonstrate.

APPENDIX A: AIR POLLUTION GENERATION FROM AREA SOURCES

Household Emissions

Residential density is highly correlated with air pollutant emission density. Figure 1 reveals an upward sloping curve of sulfur oxide emissions from the burning of coal, fuel oil, and gas as dwelling units per square mile increase. However, there are indications that various devices may allow high density situations with minimum amounts of air pollution.

The first solution that comes to mind is the use of fuels such as an oil fuel without sulfur, which gives off a considerably smaller degree of pollutants. According to existing population densities in the St. Louis area, only the most populated areas would exceed the density requirements if all units used oil, [14]. It could be that density standards could be related to the type of fuel to be used. A study done in Northeastern Illinois related allowable densities to sulfur contents of each fuel. For coal, the maximum number of dwelling units allowed per acre would be 3.1, for No. 6 oil, 6.3, and for No. 2 oil, 13.4. Heating with natural gas results in almost no SO_2 emissions and electricity produces none. If these sources were utilized, dwelling unit densities would not have to be at all restricted for air pollution reduction. [14]

The advantage gained by utilizing "sulfur-less" fuels, could be compounded if we combined this factor with other considerations. Certain structural arrangements may produce greater amounts of pollution. Housing units so constructed should be first serviced by special fuel devices.

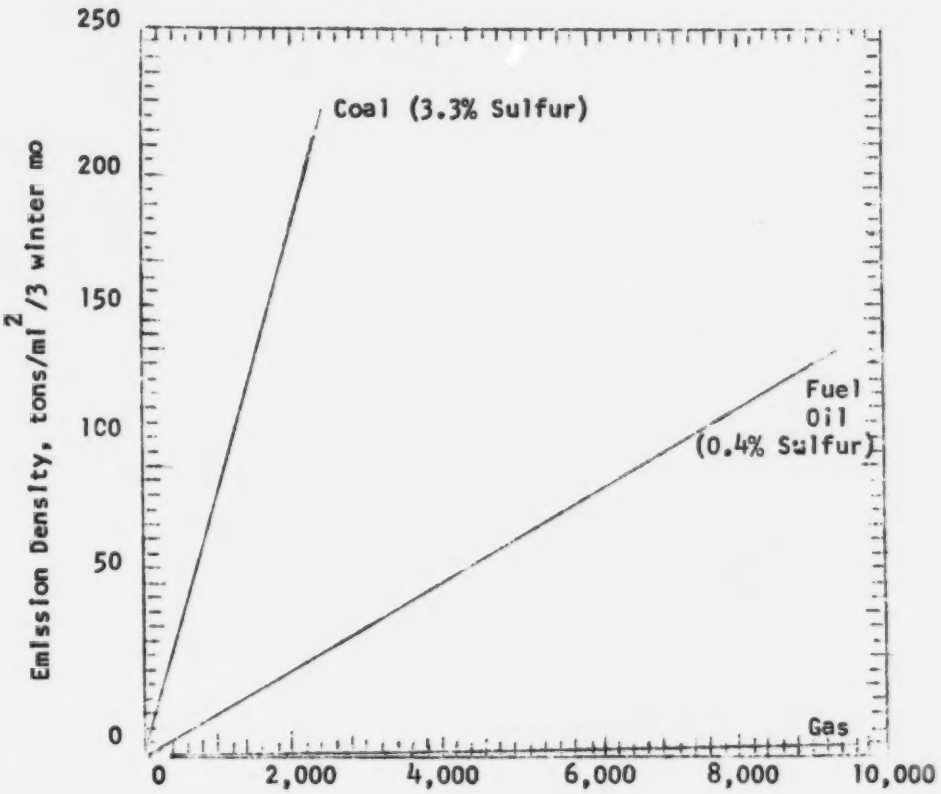


Fig. 1: Sulfur oxide emissions from burning of coal, fuel oil, and gas in residential heating units.

Source: Farmer [8; p. 50]

Of course, future builders should be aware of and avoid such configurations. It has been observed that any extensive construction has the effect of modifying winds of medium velocity. All buildings act as barriers to air currents. However, this is most valid where the horizontal profile is irregular. A uniform row of four story houses does less to interfere with air currents than does a row of low houses alternating with tall ones, (Pelle [15]). It has also been concluded that in the design of building shapes, orientations, openings, overhangs, and other architectural features there are many variables that affect the air flow around buildings, [1]. The height of a basic building block increases low pressure, turbulent downwind eddies. The larger this low pressure eddy area is, the less air swirls around building walls. Although little is known about the effect of slower currents in relation to tall buildings, it is accepted that the effect could be reduced by the design and groupings of such buildings, (Graham [10]). Le Corbusier has created a core city with triple level streets, alternating skyscrapers and parks with a very concentrated population. In his city, the pollution plume or air pocket would have a relatively short distance to travel before it left the confines of a densely populated area, (Rihm [16]). Therefore, it is not solely the macro form itself that influences the air pollutant resultant but also the composition of buildings and open space. The architect has structured a city with skyscrapers in open landscaping, allowing a high density population with an acceptable amount of air pollution.

Meteorological conditions may have a significant effect on air pollution and there is no doubt that if planners took "nature's behavior" into consideration, some degree of air pollution could be diminished

without tampering with density requirements. If a home with a warmer climate is built with a large roof overhang, the air conditioning load could be decreased. Conversely, if houses in colder areas were not built with very large windows, less fuel would be burned for heating purposes, thus less pollutants emitted, (Dixon [6]). A very plausible solution accepting the situation as it now stands would involve a close regulation of burning procedures. If concentration goes beyond a certain threshold at a certain time, one could delay the use of incinerators to shift their pollution to a more "opportune" time - i.e. when a larger quantity of air is available, (Frenkiel [9]).

It is impossible to deny that ambient sulfur dioxide levels correlate with population (r=0.72) because in many cases the use of sulfur containing fuels is directly related to the number of people in a community (r=0.84) and the quantity of sulfur-containing fuels used in a community has a direct influence on the average ambient sulfur dioxide level (r=0.85), (Faoro [7]). It is also undeniable that sparse efforts have been devoted to the correlation of urban and air pollution. The New York Department of Air Pollution has not considered the effects that population dispersal into different building configurations or into units of varying heights could have on solving the dilema. They did suggest that height was a minimal factor, offering the information that 20 story buildings do not give off twice as much smoke as 10 story buildings. Unaware of "the meteorological approach", a representative felt that a larger building may be a more optimal choice. He justified this attitude by the fact that they have a more constant flow of oil, allowing the boilers to be "regulated better".

Automobile Emissions

Double the population of a zone and air pollution generated by automobile traffic originating there doubles. This reasonable assumption made in this research needs several qualifications. Firstly, it has been observed that stop and go traffic produces more pollution. Expressways may encourage more people to drive. However, if high speeds are maintained, they may be a device that limits fumes. Vehicular speed is an important element in the control of automotive emissions. Less impurities are created as gasoline is burned more thoroughly at higher speeds. With the number of vehicles constant, sluggishly moving, stop and go traffic produces more pollution per vehicle miles than swiftly flowing traffic. Grades on highways also increase the flow of pollution by encouraging acceleration and deceleration, (Pelle [17]). Expressways, however, minimize acceleration. There is strong evidence that emissions of carbon monoxide and hydrocarbons are negatively correlated with average vehicle speed, (Mayer [12]; pp. 41-40). In addition, the uncontrolled emissions of carbon monoxide in the exhaust gases are approximately 3.1% by volume under average operating conditions. This varies from about 6% by volume at 5 miles per hour to 2% at 60 miles per hour. (Williams [17]).

Further data strengthening the negative correlation is provided by graphs indicating a decreasing manifold vacuum versus speed after a peak is reached at 30 miles per hour, (Bonamassa [5]; p. 386). The faster an engine runs the faster the exhaust or manifold valve opens and closes. The valve, if the vehicle is at high speeds is not open long enough to create a vacuum. Thus, less exhaust impurities are allowed by the manifold through the series of passages from the engine cylinder.

Thus, it is evident that it is not solely the number of automobiles that multiply the pollution problem. If vehicles could be kept at a constant and fairly rapid speed less pollutants would be emitted. What is the answer to this proposal? A study done by the Northeastern Illinois Planning Commission recommends the orientation towards major expressway systems by locating industrial parks, shopping centers, apartments, and office buildings close to expressway ramps. It may be possible to build "cross-town" expressways to divert traffic away from critical areas also allowing higher speeds to be maintained. The actual construction of expressways in conjunction with the location of populated districts may limit the amounts of pollution focused on a given area. If freeways are oriented along the wind, they will result in high concentrations of pollutants near their downwind end. The concentration for constant source strength and steady wind increases as the square root of the distance downwind. If a line source is oriented across the wind direction, the concentrations are greatest close to the source, (Nelburger [13]). Another construction device which would at least help to absorb some of the noxious pollutants would be the proliferation of green buffer strips along the freeways. This discussion does not attempt to debate the varying social costs of the alternatives proposed. Nonetheless, it should be noted that the location of multiple expressway systems will be sure to intrude on a significant portion of the population whose interests must not be overlooked in the rush for a utopia of constant speed.

It is evident that when traffic concentration is high, the carbon monoxide concentration rises. A study done of carbon monoxide concentration in the Detroit, New York, and Los Angeles area revealed that concentration

reached its peaks in the morning and late afternoon. The correlation coefficients between carbon monoxide and the traffic count ranged from 0.75 to 0.95, (Bergman [3]). This dilema may be resolved without scattering the population. Concentration may be reduced in two ways. The quantity of pollutant can be reduced or the quantity of air increased, (Battan [2]). Although the following solution is not related to urban form it involves a restructuring of the urban scene or pattern. A manipulation of traffic patterns, i.e. shortening the workday to 6 or 7 hours by beginning it at 10 A.M. during the summer may lessen the contamination. Presently, traffic is heaviest at 7 A.M. in Los Angeles and auto emissions are mixed through a very small stmospheric volume yielding a high rate of reaction which forms photochemical smog. By 10 A.M., the mixing volume would be about seven times greater and the pollutant concentrations correspondingly lower. If the returning home traffic peak occured before 5 P.M., oxidant concentrations would be up to 54% lower than now observed, (Kauper [11]). Another approach would be less complex. It would involve improving traffic flow at given levels of traffic by devices such as better traffic control, redesigned intersections and traffic bottlenecks, automatic control devices to govern individual vehicles (automated highway), and better system labeling. "For suitable types of expressways and highway facilities, controlling vehicle entry at access points may prevent shock waves and capacity reductions which in turn lead to increased delays and lowered speeds." (Blum [4]).

A number of approaches not related to the urban structure at all could also be implemented. These include drives to promote better driving habits to get people to follow more efficient driving cycles, attempts to

cause polluting vehicles to be operated more efficiently, taxes related to emissions, and the use of rented or leased vehicles or taxis which are often kept in better repair. (Blum [4]).

Thus, it is clear that a high population density resulting in increased automobile traffic does not have to necessarily cause a startling rise in air pollution. If methods are employed to lessen traffic congestion, encourage the use of mass transit facilities and expressways, and wisely situate the freeways, the link between people, automobiles, and pollution may be weakened.

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APPENDIX B: AIR POLLUTION DIFFUSION MODEL

INTRODUCTION

For a review of air pollution models from the perspective of urban planning see Savas [3] pp. 168-170. The major distinction is between micro-scale models applicable in street "canyons" and macro-scale models applicable to whole urban areas. Micro-scale models are very underdeveloped, while several macro-scale models exist. The macro-model used in this paper's experiments is a simplified version of those found in Turner [4] and Miller and Holzworth [2].

THE MODEL

After dividing an urban area into grid cells and recording the wind rose, a diffusion model is needed to describe how much pollution emitted in one cell reaches the other cells. For this research's experiments an urban area of thirteen grid cells is assumed and the wind is assumed to blow from the north-west fifty percent of the time and from the south-west fifty percent of the time. (See Figure 1.) Atmospheric conditions are assumed to approximate stability class three. (See Turner [4] p. 91.)

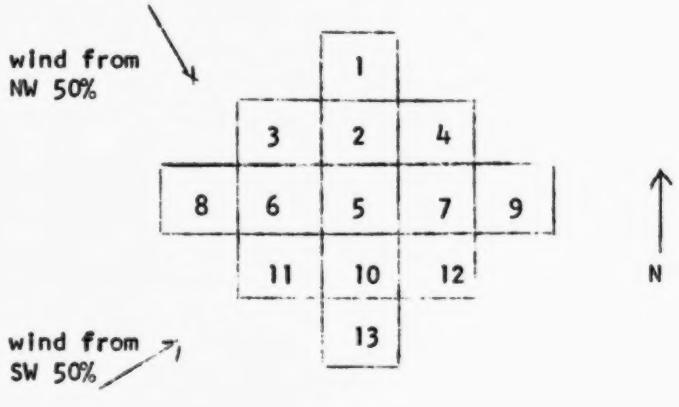

Figure 1: Urban Area Divided into Grid Cells

With an urban area characterized by a vector of grid cells, the diffusion model takes the form of a linear transformation of emission rates by grid cells into concentration by grid cell. The model used in this research is presented by the matrix, D, in Figure 2. Post multiplication by this matrix transforms a vector of pollutant emission rates into a vector of pollutant concentrations.

Since urban form considerations depend only upon the pattern of air pollution distribution, the diffusion model has been scaled to make the impact of a cell upon itself equal to one. The impacts of a cell's emissions upon other cells are given by the off diagonal elements in the diffusion matrix. For example, pollution emitted in cell 1 produces in cell 2 approximately 2% of the concentration produced in cell 1; and pollution emitted in cell 7 produces in cell 9 approximately 4% of the concentration produced in cell 7.

DERIVATION OF THE MODEL

Following the diffusion literature, Gifford [1] for example, assume that pollution from a point source is normally distributed in the cross-wind direction with a standard deviation σ_y meters, independently normally distributed in the vertical direction with a standard deviation σ_z meters, and transported by the wind at μ meters per second. Then relative concentration at a point (x,y,z) is given by:

$$(1) \qquad \frac{X(x,y,z)}{Q} = \frac{1}{2\pi\sigma_y\sigma_z\mu} \exp\left[-\frac{1}{2}\left[\frac{y^2}{\sigma_y^2} + \frac{z^2}{\sigma_z^2}\right]\right]$$

where y is measured along the cross-wind direction, z in the vertical direction, and x in the along-wind direction from the point source. Integrating over an infinite line source gives the following formula for

Linear Transformation
From Pollution Emission by Cell

To Pollution Concentration by Cell	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1.0	.02	.034			.006		.017					
2	.02	1.0	.04		.02	.034		.006			.006		
3			1.0			.02		.034					
4	.034	.04		1.0	.035	.006	.02			.006	.017		
5		.02	.034		1.0	.04				.02	.034		
6			.034			1.0		.04			.02		
7	.006	.035	.005	.02	.04		1.0			.034	.006	0.2	.006
8								1.0					
9	.017	.005		.034			.04		1.0	.006		.034	.017
10			.006		.02	.034		.006		1.0	.04		.02
11						.02		.034			1.0		
12		.006	.017		.034	.006	.02			.04		1.0	.034
13						.006		.017		.02	.035		1.0

Figure 2: Air Pollution Diffusion Matrix, D, (Stability Class Three; Wind 50% NW, 50% SW).

downwind relative concentration:

$$(2) \quad \frac{X(x,z)}{Q} = \int_{y=\infty}^{\infty} \frac{X(x,y,z)}{Q} dy = \frac{1}{[2\pi]^{0.5}\,\sigma_z \mu} \exp\left[\frac{-z}{2\,\sigma_z^{\,2}}\right]$$

Finally, setting z = 0 assuming that pollution is emitted and received at ground level, and multiplying by two to account for reflection of pollution off the ground, yields the following formula for the relative concentration at a point x meters downwind from a line source:

$$(3) \quad \frac{X(x)}{0} = \frac{0.8}{\sigma_z \mu}$$

Empirical investigations summarized graphically in Miller and Holzworth [2] p. 46, Turner [4] p. 85, and Gifford [1] p. 11, have shown that the vertical standard deviation, σ_z, is an increasing function of distance from the source. In addition, σ_z depends upon atmospheric stability which in turn depends upon solar radiation and wind speed. For the purposes of this paper's calculations we assume that:

$$(4) \quad \sigma_z = [0.41]\,[x/\mu]^{0.92}$$

which is the equation Miller and Holzworth used for Turner's [4], p. 91, stability class three.* Substituting (4) into (3) yields a formula for downwind relative concentration of:

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* Equation (4) was received in a letter from Miller on May 18, 1967. For a graphical presentation of this relationship see Miller and Holzworth [2] p. 46.

(5) $\frac{X(x)}{Q} = k\, x^{-.95}$; where $k = [.195]\mu^{-.08}$

Assume that each grid cell has uniformly distributed emissions. When the wind crosses a grid cell diagonally (see Figure 3) the average relative

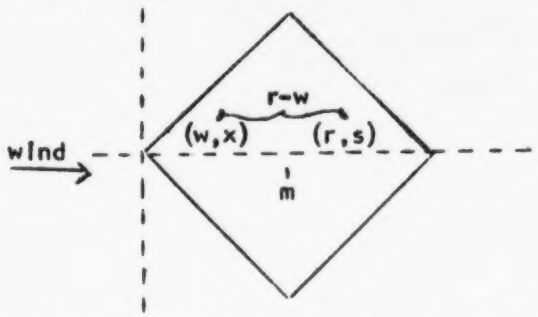

Figure 3: Diagram for Equation (6)

concentration of pollution in the cell is given by:

(6) $$\frac{1}{m^2} \int_{s=0}^{m} \int_{r=s}^{2m-s} \int_{w=s}^{r} k[r-w]^{-.92}\, dw\, dr\, ds$$

where m is one-half the diagonal distance across the grid cell in meters. The point (r,s) is an arbitrary reception location. The point (w,s) is an arbitrary upwind emission location. The distance between these two points is r-w. The integration with respect to w sums the impact of all upwind sources on (r,s). The integrations with respect to r and s sum the receptions of all points in the grid cell. Due to symmetry the integration is only performed over one-half of the grid cell, so dividing by m^2 gives average relative concentration.

The quantity calculated by equation (6) is the concentration in a grid cell caused per unit emission in the same cell. This quantity belongs in the diagonal spaces of the air pollution diffusion matrix. However, since

matrix model has been scaled by setting the diagonal elements equal to one, the quantity in (6) is used to divide the average concentrations calculated for the other cells.

The average relative concentration in a grid cell directly downwind from a source cell (see Figure 4) can be calculated by:

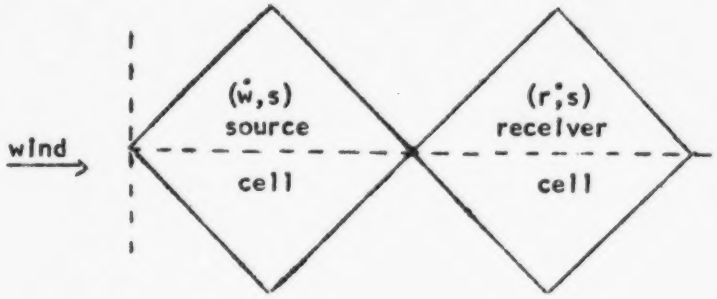

Figure 4: Diagram for Equation (7)

$$(7) \quad \frac{1}{m^2} \int_{s=0}^{m} \int_{r=2m+s}^{4m-s} \int_{w=s}^{2m-s} k\,[r-w]^{-.92}\, dw\, dr\, dw$$

The ratio of (7) to (6), found by the obvious but laborious calculations, is 0.067. This means that the average concentration caused by the source cell in the downwind cell is 6.72% of the average concentration in the source cell. The fact that this ratio does not depend on wind speed or cell size means that the model, and the urban form conclusions following from it, are not sensitive to these factors.

Proceeding in the fashion of equations (6) and (7) one can calculate the average concentrations in all the downwind cells, stated as fractions of the average concentration in the source cell. The results of these calculations are displayed in Figure 5. Cells with approximately zero concentrations are not shown.

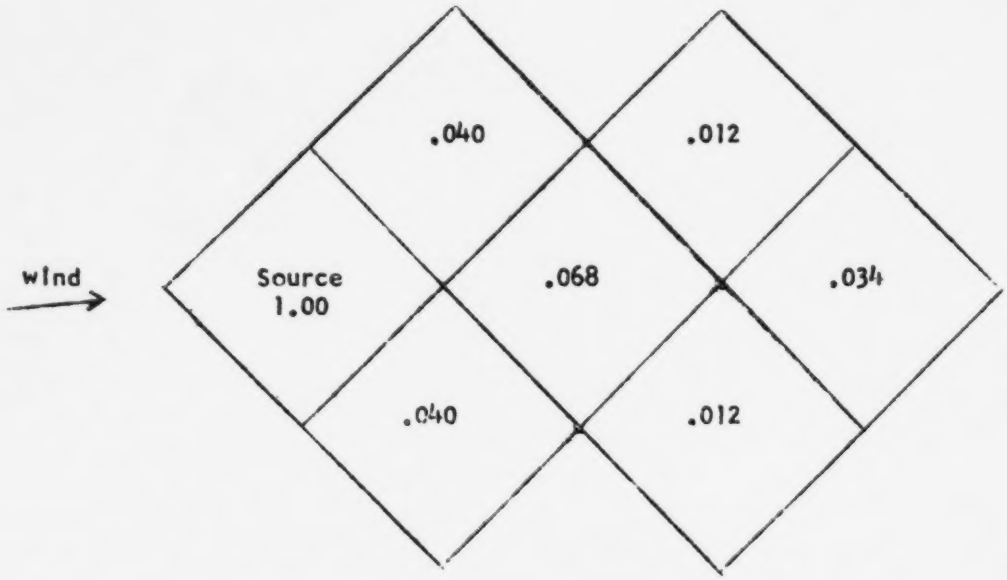

Figure 5: Average Concentrations Caused in Downwind Cells by Source Cell, Stated as Fractions of the Average Concentration in the Source Cell.

Using the diffusion information recorded in Figure 5, diffusion matrices for winds from the northwest and the southwest are easily constructed. Then adding these two matrices and dividing by two, produces the diffusion matrix, D, in Figure 2, and completes the derivation of this diffusion model.

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APPENDIX C: EVALUATING AIR POLLUTION DAMAGE

The Haber Law

In this paper damage caused by air pollution has been evaluated according to the Haber Law of gas poisoning: cost = concentration times duration.* Only scattered evidence supports application of the law to air pollution. Estimates have been made of damage to property, such as agriculture, fabrics, rubber, electrical contact points and paint with varying pollution concentrations. See, for example, Kneese [8], p. 7 and Interstate Air Pollution Study Vol. VI, [1], pp. 35-49. Ridker [12] applied least square regression methods to relate pollution concentration to actual real estate costs in St. Louis, but this model cannot be applied elsewhere because its form and content are based largely on data specific to the time and city studied. Knowledge of the costs of ill health associated with air pollution is in a worse state than of other costs. The most commonly used measure of health costs appears to be the capitalized value of gross production lost. To be accurate evaluation of costs should include the psychic costs of loss of health and life and the costs of constant minor pain that result from exposure to air pollution. Silver [14] is developing a system for health measurement based on airplane design costing which is sensitive to these costs, but the dimensions in his system require refinement before they can be applied.

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* Fritz Haber formulated this law as concentration x time = ct or lethal dose, to determine which gases were most deadly for use in World War I. (See Wachtel [18].) It is time the formula was put to humane use.

There is evidence to support the Haber law of gas poisoning as a function of air pollution costs for some gases for periods of several minutes to several hours. Many studies support this relationship as an expression of the impact of carbon monoxide (See Interstate Air Poilution Study Vol. VI [1] and Silver [15]), but no systematic study has been undertaken to verify the range of concentrations of carbon monoxide and durations for which the relationship holds. For other gases much less evidence is available in which time of exposure is used as a variable. It has been shown that exposure to greater than average concentrations of sulfur compounds in the air relates positively with high morbidity and mortality rates in persons already suffering respiratory diseases. (Interstate Air Poliution Study, Vol. VI [1], pp. 6-12.) Most of these studies were based on examination of hospital records during episodes of high levels of these pollutants. They did not record the actual concentration or duration of exposure, nor even the intensity of the affected person's response. Closer study is needed to understand the relationship among these factors.

Damage Thresholds

The Haber law of gas poisoning was formulated to express experience in the laboratory with discrete gases in constant quantities over short periods of time. Chemicals which pollute the air often act in a variety of ways outside the experimental assumptions upon which the Haber law is based. For example, some pollutants appear to exhibit damage thresholds. Also, there is evidence that the damage function is modified when pollutants are experienced intermittently, in combination or over extended periods of time. In our work with pollutants we must keep in mind the various

kinds of damage relationships exhibited by pollutants under some circumstances which modify and sometimes contradict the Laber law. In the following paragraphs we shall examine evidence of these other relationships.

Fechner postulated that below certain levels a gas can be tolerated without any alteration in perception or functioning of the organism. (See Prentiss [10], p. 13.) Modified by Fechner, the Haber law is stated: cost = (concentration - threshold) x time. Research efforts have found widely varying thresholds for different gases. Henderson and Haggard [7] found a threshold of 300 p.p.m.-hrs. for carbon monoxide. A Harvard School of Public Health research team found no change in humans subjected to sulfur dioxide levels up to one to two p.p.m. for brief periods. (Prindle [11], p. 902.)

Temporal Variations: Continuous Exposure

Continuous exposure to low levels of air pollutants may cause damage disproportionately higher than the Haber law would predict. This kind of exposure results in chronic minor ailments which are seldom reported in medical records. Brief concentrated exposures precipitate acute seizures in chronic sufferers. It is difficult to isolate the effects of the concentrated episode from those of continuous exposure because the effects of the latter are reported only during times of concentrated exposure. The following examples illustrate this kind of situation.

In New Orleans the average admission of asthmatics to Charity Hospital was 25 per day between 1953 and 1961. Frequently, however, outbreaks of asthmatic attacks involved four to eight times as many patients. It was found that the outbreaks were related to particles of a silicon-containing

compound emitted into the atmosphere as a result of poor combustion at the city dumps. This could be the case of an air pollutant acting as an allergen creating an allergic response in persons made sensitive by chronic levels of pollutants. ("Six Years of Research in Air Pollution, July 1, 1955-June 3, 1961" [2].) Similar high asthma treatment rates were found in Nashville during episodes of high sulfur dioxide content in the air. (Zeidberg et al, [19].) In Yokohama, Japan, high incidence of a condition called "yokohama asthma" was reported among American servicement and their dependents. When sufferers left the area, they no longer experienced asthmatic attacks. Others who were not quickly evacuated continued to suffer asthmatic symptoms. Because this condition appears to be truly a chronic effect of contamination in the atmosphere, it is not referred to as "air pollution asthma". (Phelps [9].) It is not known what specific pollutants cause this condition nor is the exposure period necessary for damage determined. Studies on migrants from Britain to South Africa and New Zealand also suggest the role of chronic air pollution as a causative factor in lung cancer. (Dean [4] and Eastcott [6].) In the United States a high degree of correlation has been found between respiratory diseases lasting at least eight days and high levels of sulfate pollution. (Dohan [5].) If the case histories of those affected and the levels and periods of exposure were more completely documented, this might partly explain the striking rise in the prevalence in respiratory diseases which has been observed recently. (See Prindle [10] for a summary of these kinds of findings.)

Throughout urban America today we are exposed continuously to low levels of various atmospheric pollutants. We have scattered data that show that

these pollutants can be harming our health. Intensive study is needed of the effects of long term exposure to low levels of pollutants before we can know the actual impact of these gases on our health. We must consider our standards for air pollution levels provisional until we know more about the impact of continuous exposure to low concentrations of pollutants on human health.

Temporal Variations: Intermittant Exposure

There is evidence that intermittant exposure to a pollutant will produce less serious effects than constant exposure to the same concentration of the gas. It has been shown that eight hour exposure to 30 p.p.m. of carbon monoxide will cause "alteration of bodily function or lead to chronic disease". (See "Technical Report of California Standards for Ambient Air Quality and Motor Vehicle Exhaust", [3] especially p. 15.) Yet police who patrol the Holland Tunnel experience two hours on, two hours off, two hours on again exposure to 70 p.p.m. of carbon monoxide and show little evidence of damage, (Sievers [13]) perhaps because the body is given a chance to repair between sessions. The impact of intermittant exposure to pollutants and the human body's response must be verified by extensive and systematic research. Pollutant levels in ambient air to which most people are exposed fluctuate widely and to some extent rhythmically. If recovery periods are significant, it is possible that we could reduce the deterioration of our health by changing the rhythm of exposure!

Non-additivity

In this paper damage by pollutants is assumed to be additive. Although this is the case for pollutants which do not react with others, there is increasing evidence that interactions among many gases result in a variety

of non-additive effects upon humans. The most widely known product of such reactions is photochemical smog.

The photochemical reaction occurs when sulfur dioxide, nitrogen dioxide and aldehydes absorb ultraviolet radiation. The rate of reaction depends upon the compount involved, as do the products, and therefore the damage consequences of the reaction.

SO_2 + ultraviolet radiation $\rightarrow$ ozone + SO_3 + H_2SO_4

NO_2 + ultraviolet radiation $\rightarrow$ ozone $\rightarrow$ NO_2 or nitric acid

aldehyde + ultraviolet radiation $\rightarrow$ ozone + other compounds

Everything to the right of the first arrow is considered smog and causes harmful effects to humans. Smog has a feedback effect on the rate of its own production. The more smog hanging over a city, the less ultraviolet radiation can get through to provoke photochemical reactions. It has been estimated that as much as 50% of the ultraviolet radiation can be screened by smog.

Obviously the impact on humans of gases which take part in photochemical reactions is much more complex than can be expressed by a simple sum of cost equations.

Widespread inherent protection from acute response to oxidants can result from inhalation of a single oxidant air pollutant. Tolerance to multilethal doses of ozone has been produced in rats by low concentration (.3 p.p.m. in mice) exposures for short durations (a few hours). (See Stokinger [16], p. 316.) Cross-tolerances to lethal doses of ozone have been produced by exposure to ketene, nitrogen dioxide, and hydrogen peroxide. (See Svirbely and Stokinger's unpublished results, in

Stokinger [16], p..316.) Tolerances to oxidants have been caused by other substances as well--e.g. by oil mists and rat poisons in rats. The mechanisms by which this oxidant tolerance develops is at present elusive. (See Stokinger [16], pp. 319-321.)

Abnormally great damages as a result of reactions among pollutants have been studied less. Svirbely and Stokinger [17] found that concentrations above 1.5. p.p.m. of hydrogen peroxide inhaled with non-injurious concentrations of ozone (ca. one p.p.m.) were lethal for some mice although more than 200 p.p.m. of hydrogen peroxide alone are required to produce death in the tested species.

Most of the gases involved in these non-additive reactions are available in polluted urban air or as a result of photochemical reactions. Further study of non-additive reactions among gases can aid us in minimizing the damage caused by pollutants in the air.

Conclusion

The state of knowledge of damage functions of the various pollutants is uneven. A large portion of chemical pollutant research has been devoted to the specific physiological effects of carbon monoxide, to the relationship between hospital admission rates and sulfation levels, and to the reactions among constituents of photochemical smog. The physiological effects of sequential and simultaneous exposure to various chemical pollutants are being studied by a few researchers, as are the effects of dustfall on respiratory health and a variety of other specific pollution-related problems.

Thorough knowledge of the ways air pollutants cause physical damage and ill health is essential because researchers need a measure for deciding where the various pollutants stand in priority for attention. Government officials need knowledge of damage caused by pollution as an indication of air pollution's place in the hierarchy of national problems, which in turn will determine the proportion of research and other funds to allocate for it. The public should be given some overall perspective concerning the seriousness of the problem.

A systematic determination of the damage caused by all atmospheric pollutants controlled for all naturally occurring conditions of ambient air is needed. Silver's health measurement system offers promise of means to compare damage costs once they are determined. We must develop such a means of measuring pollution damage before we can co-ordinate our efforts and control the damage caused by air pollution.

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APPENDIX D: EXPLAINING URBAN POPULATION DISTRIBUTION BY DENSITY COST FUNCTIONS

Clark [1] was the first to note that urban population densities tend to decline negative exponentially as distance from the urban center increases. Ever since his seminal observation economists, ecologists, geographers, and city planners have been fascinated by the simplicity of the negative exponential pattern. Much work has been done on refining and testing Clark's hypothesis. (See Swerdloff [3]) for a recent effort. Work has also been done on explaining the forces behind the observed density structure. See, for example, Muth [2].

The present formulation "explains" the negative exponential decay in population density in terms of two competing forces. First, transportation costs to the urban center provides pressure for population concentration. Second, preferences for low density environments and construction cost considerations cause an opposing pressure for population dispersion.

Notation

t = total population to be distributed

R = radius of urban area

x = distance from center of urban area

$f(x)$ = population density as a function of distance from the center of the urban area.

Assumptions

(1) The total cost during some unit of time of transporting people to and from the urban center is given by:

$$A \int_{\theta=0}^{2\pi} \int_{x=0}^{R} [x\, f(x)]\, x\, dx\, d\theta$$

where A is some constant. This expression assumes that the per person transport cost is directly proportional to distance, and that all people have the same frequency of trips to the urban center.

(2) The total cost, during the same unit of time, of living at high densities is given by

$$B \int_{\theta=0}^{2\pi} \int_{x=0}^{R} [f(x) \log f(x)]\, x\, dx\, d\theta$$

where B is some constant. This expression assumes that individual ammenity cost of high density environments increase less than proportionately with density. Specifically, the cost is assumed to increase as the logarithm of density.

Derivation of a Negative Exponential Density Pattern

The residential pattern, f(x), that minimizes the sum of the transportation costs and the density costs must minimize

$$\int_{\theta=0}^{2\pi} \int_{x=0}^{R} [A x^2 f(x) + B x f(x) \log f(x)]\, dx\, d\theta$$

$$= 2\pi \int_{x=0}^{R} [A x^2 f(x) + B x f(x) \log f(x)]\, dx$$

Subject to the constraint that

$$t = \int_{\theta=0}^{2\pi} \int_{x=0}^{R} f(x)\, x\, dx\, d\theta$$

$$= 2\pi \int_{0}^{R} f(x)\, x\, dx$$

Applying the Euler equation of the calculus of variations to this constraint minimization problem we obtain the necessary condition

$$\frac{\partial}{\partial f} [A x^2 f(x) + B x f(x) \log f(x) + \lambda x f(x)] = 0$$

Performing the indicated differentiation leads to:

$$\log f(x) = -\frac{A}{B} x - \frac{[B + \lambda]}{B}$$

Finally, taking the antilog, we conclude that our assumptions imply a negative exponential density pattern:

$$f(x) = C e^{-\frac{A}{B}x}$$

Derivation of a Linear Density Pattern

Muth [2; p. 214], in his statistical analysis of 46 United States cities, reaches the conclusion that "... the negative exponential function is the best simple approximation to the pattern of population density decline with distance from the urban center in urban areas". However,

Muth found that the fit of the negative exponential function is far from perfect. For an analysis that only wishes to recognize that population density declines with distance from the urban center, even the above simple model is overly complicated. A simpler model results if the expression in assumption (2) is replaced by:

$$B \int_{\theta=0}^{2\pi} \int_{x=0}^{R} [f(x)]^2 \, x \, dx \, d\theta$$

where B is some constant. In this alternative assumption, the individual amenity cost of density is assumed to be directly proportional to density. With this assumption the Euler equation for cost minimizing population density becomes

$$Ax^2 + 2Bx f(x) + \lambda x = 0$$

which leads us to the conclusion that population density now declines linearly with distance from the urban center.

$$f(x) = C - \frac{A}{2B} x$$

Conclusion

Assuming that transport costs are directly proportional to distance, and that all families have an equal propensity to travel to the city center, the observed pattern of urban residential densities can be explained by postulating a density cost function, and then minimizing the total of transport and density costs. The assumption that density cost is proportional to the logarithm of density leads to a negative exponential density function. The assumption that density cost is directly proportional to density leads to a linear density function.

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APPENDIX E: CALCULATING THE POPULATION DISTRIBUTION THAT MINIMIZES A QUADRATIC COST FUNCTION

The problem under consideration is finding the spatial distribution of households, $\underline{p}$, that minimizes a quadratic function of air pollution costs, $\frac{1}{2}\,\underline{p}'\,A\,\underline{p} + \underline{b}'\,\underline{p}$, where A is a symmetric matrix. This minimization is subject to the constraint that the population districted must sum to the total population of the urban area, $\underline{j}'\underline{p} = t$. It is also subject to the constraint that negative population are impossible, $p \geq 0$. Assuming that these non-negativity constraints do not bind,* we start our search for a solution by introducing the Lagrange multiplier, λ, so that the problem becomes one of finding the population vector, $\underline{p}$, and the multiplier λ, that minimize

$$f(p, \lambda) = \tfrac{1}{2}\,\underline{p}'\,A\,\underline{p} + \underline{b}'\,\underline{p} + \lambda[t - \underline{j}'\,\underline{p}]$$

First Order Conditions

The first order conditions for minimizing this expression are obtained by setting first derivatives of $f(\underline{p}, \lambda)$ equal to zero:

$$\frac{\partial f(\underline{p},\lambda)}{\partial \underline{p}} = A\,\underline{p} + \underline{b} - \lambda \underline{j} = 0$$

$$\frac{\partial f(\underline{p},\lambda)}{\partial \lambda} = t - \underline{j}'\,\underline{p} = 0$$

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* If this assumption is incorrect, negative populations will be assigned to areas, and the erroneous assumption will be all too apparent.

The solution to this set of simultaneous equations gives the necessary conditions for $\underline{p}$ and λ to be a stationary point of $f(\underline{p}, \lambda)$:

$$\lambda^* = \frac{t + \underline{j}' A^{-1} \underline{b}}{\underline{j}' A^{-1} \underline{j}}$$

$$\underline{p}^* = \left[\frac{t + \underline{j}' A^{-1} \underline{b}}{\underline{j}' A^{-1} \underline{j}} \right] A^{-1} \underline{j} - A^{-1} \underline{b}$$

Second Order Conditions[1]

For this population distribution, $\underline{p}^*$, to provide a global minimum to $f(\underline{p}, \lambda)$ it is necessary that

$$f(\underline{p}, \lambda) - f(\underline{p}^*, \lambda) > 0; \text{ for all } \underline{p} \neq \underline{p}^*, \underline{j}'\underline{p} = t$$

Evaluating the function, f, at $\underline{p}$ and $\underline{p}^*$ this requirement becomes

$$\tfrac{1}{2} \underline{p}' A \underline{p} - \tfrac{1}{2} \underline{p}^* A \underline{p}^* + [\underline{b} - \lambda \underline{j}]' \underline{p} - [\underline{b} - \lambda \underline{j}]' \underline{p}^* > 0$$

When this is rewritten as

$$\tfrac{1}{2} [\underline{p} - \underline{p}^*] A [\underline{p} - \underline{p}^*] + [\underline{b}' - \lambda \underline{j}' + A \underline{p}^*]' [\underline{p} - \underline{p}] > 0$$

We can use the first order condition $\underline{b}' - \lambda \underline{j}' + A \underline{p}^* = 0$ to reduce the second order condition to

$$[\underline{p}' - \underline{p}^*] A [\underline{p} - \underline{p}^*] > 0; \text{ for all } \underline{p} \neq \underline{p}^*, \underline{j}' \underline{p} = t$$

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[1] The argument in this section is adapted from Boot [3], pp. 25-28.

From this formulation we see that if A is a positive definite matrix then a global minimum to our problem is assured. Conversely, if A is negative definite, our first order conditions are maximizing the air pollution cost function.

Should A be an indefinite matrix, however, a further investigation of second order conditions is called for. In arriving at a second order condition in terms of the matrix A we did not use the information that $\underline{j}' \underline{p} = t$. Adding this constraint leads to a less demanding second order condition.

Using the following partitions

$$p = \begin{bmatrix} p_1 \\ \underline{p}_2 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \underline{a}' \\ \underline{a} & A_{22} \end{bmatrix}$$

the second order condition can be written in a form that isolates the first component of the population vector

$$\begin{bmatrix} p_1 - p_1^* & \\ \underline{p}_2 & \underline{p}_2^* \end{bmatrix}' \begin{bmatrix} a_{11} & \underline{a}' \\ \underline{a} & A_{22} \end{bmatrix} \begin{bmatrix} p_1 - p_1^* & \\ \underline{p}_2 & \underline{p}_2^* \end{bmatrix} > 0$$

Then, partitioning $\underline{j}$:

$$\underline{j} = \begin{bmatrix} 1 \\ \hline \underline{j}_2 \end{bmatrix}$$

and using the constraint $\underline{j}' \underline{p} = t$, the first component of the population vector can be written in terms of the other components

$$p_1 = t - \underline{j}_2' \underline{p}_2$$

$$p_1^* = t - j_2 \underline{p}_2^*$$

Substituting these values for p_1 and p_1^* into the partitioned second order condition the requirement becomes

$$[\underline{p}_2 - \underline{p}_2^*]' [a_{11} \underline{j} \underline{j}' - 2 \underline{a} \underline{j}' + A_{22}] [\underline{p}_2 - \underline{p}_2^*] > 0;$$

$$\text{for all } \underline{p}_2 \neq \underline{p}_2^*$$

From this statement of the second order condition we see that a global minimum is guaranteed if the matrix $[a_{11} \underline{j} \underline{j}' - 2 \underline{a} \underline{j}' + A_{22}]$ is positive definite.

Summary of Second and First Order Conditions

If either A or $[a_{11} \underline{j} \underline{j}' - 2 \underline{a} \underline{j}' + A_{22}]$ is a positive definite matrix, then

$$\underline{p} = \begin{bmatrix} \dfrac{t + \underline{j}' A^{-1} \underline{b}}{\underline{j}' A^{-1} \underline{j}} \end{bmatrix} A^{-1} \underline{j} - A^{-1} \underline{b}$$

minimizes $\underline{p}' A \underline{p} + \underline{b}' \underline{p}$ subject to the constraint $\underline{j}' \underline{p} = t$.

Checking that a Matrix is Positive Definite

Since the second order condition for our air pollution minimization problem requires that a matrix be positive definite, it is necessary to be able to recognize positive definiteness when it occurs. Our first two criteria are the standard theoretical tests, the third is a general operational test, and the fourth criterion applies in the special case of matricies with dominant diagonals.

Criterion 1: A necessary and sufficient condition for a symmetric matrix to be positive definite is that all its characteristic roots be positive. (Bellman [2], p. 54.)

Criterion 2: For a symmetric matrix to be positive definite it is necessary and sufficient for all the principal minors to be positive. (Bellman [2], p. 74.)

Criterion 3: A necessary and sufficient condition for a symmetric matrix to be positive definite is that all of the pivot elements in the Gauss-Jordan inversion of the matrix be positive. (Almon [1], p. 101.)

Criterion 4: If a symmetric matrix has a dominant diagonal that is positive then the matrix is positive definite. The matrix A is said to have a dominant diagonal if $\left| a_{jj} \right| > \sum_{i \neq j} \left| a_{ij} \right|$ for all j. (McKenzie [5], p. 49.)

Quadratic Forms with Dominant Diagonals

In minimizing $\frac{1}{2} \underline{p}' A \underline{p} + \underline{b}' \underline{p}$, subject to $\underline{j}' \underline{p} = t$ and $p \geq 0$, the Lagrange multiplier technique leads to the conclusion that $\underline{p} = A^{-1} [\lambda \underline{j} - \underline{b}]$ where $\lambda = [t + \underline{j}' A^{-1} \underline{b}] / \underline{j}' A^{-1} \underline{j}$. However, this conclusion is correct

only if two requirements are satisfied:

(1) the first order condition $\underline{p} = A^{-1}[\lambda \underline{j} - \underline{b}]$ must be determining a global minimum; and

(2) the non-negativity constraints must not bind, i.e. $A^{-1}[\lambda \underline{j} - \underline{b}] \geq 0$.

In our problem A is a non-negative, symmetric matrix. If A also has a dominant diagonal, $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$ for all j, then requirement 1 is assured by A being positive definite. If A has a dominant diagonal with all the diagonal elements the same, then requirement 2 is assured for a sufficiently small $\underline{b}$ vector since $A^{-1}\underline{j} > 0$.

To see that a non-negative matrix with a dominant diagonal is positive definite, start by considering $A - sI$ where s is a characteristic root of A. Assume s has a non-positive real part. Then $|a_{ii} - s| \geq |a_{ii}|$ for all i. This fact together with A's dominant diagonal, $a_{ii} > \sum_{i \neq j} a_{ij}$ for all i, implies $|a_{ii} - s| > \sum_{i \neq j} |a_{ij}|$ for all i, and therefore $A - sI$ is a dominant diagonal matrix. But all dominant diagonal matricies are non-singular. So s can not be a characteristic root if it has a non-positive real part. Therefore all the charactersitic roots of A are positive and a dominant diagonal implies positive definiteness. (McKenzie [5], p. 49.)

The proof that $A^{-1}\underline{j} > 0$ when A is non-negative, symmetric, and has a dominant diagonal with a uniform element also gives us a short-cut method of calculating A^{-1} under these circumstances. Let d be the uniform diagonal element of A. Then $A = d[I + R]$ if we define $R = \frac{1}{d} A - I$. Because A has a dominant diagonal, and since R is the residual after dividing A by the common diagonal element and subtracting the identity matrix, we

know that all the column sums of R are less than one. This fact, together with an argument by Hadley [4], pp. 118-119, enables us to conclude that:

$$\lim_{n\to\infty} R^n = 0$$

Then, since

$$[I + R]\,[I - R + R^2 - R^3 + \ldots(-R)^n] = I + (-R)^{n+1}$$

we conclude that

$$[I + R]^{-1} = I + \sum_{n=1}^{\infty} (-R)^n$$

Recalling that $A = d\,[I + R]$ we conclude that

$$A^{-1} = \frac{1}{d}\left[I + \sum_{n=1}^{\infty} (-R)^n\right]$$

or, factoring out $[I - R]$,

$$A = \frac{1}{d}\left[I + \sum_{n=1}^{\infty} R^{2n}\right][I - R]$$

Since A is symmetric, R is symmetric; so since all the column sums of R are less than one, all the row sums of R are also less than one. Therefore, $[I - R]\,\underline{j} > 0$, which enables the conclusion that $A^{-1}\,\underline{j} > 0$.

Note that when R^n converges to the null matrix rapidly enough to make $A^{-1} = \frac{1}{d}[I - R]$ a good approximation we have a very easy method of calculating A^{-1}.

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